PERGAMON

International Journal of Heat and Mass Transfer 41 (1998) 3177–3189

International Journal of HEAT and MASS TRANSFER

Model reduction in linear heat conduction: use of interface fluxes for the numerical coupling

D. Petit[a,*], R. Hachette[b]

[a] *Laboratoire d'Études Thermiques CNRS UMR 6608, ENSMA BP 109, 86960 Futuroscope Cedex, France*
[b] *Institut Universitaire des Systèmes Thermiques Industriels, CNRS UMR 6595 Technopôle de Château-Gombert, 5 Rue Enrico Fermi 13453 Marseille Cedex 13, France*

Received 26 May 1997; in final form 6 February 1998

Abstract

In the framework of numerical modelling in transient heat conduction, it is shown in this paper, how to introduce different kinds of coupling on a part of the domain boundary without changing the original matrices of the model. The technique uses specific fluxes applied to the boundary which is treated: at each time step the coupling variables—temperatures and fluxes—are computed at first, and then, the influence of the fluxes are distributed on the rest of the domain and no iteration procedure is used. This permits the treatment of unexpected problems such as: variations of heat transfer coefficients, model connection and radiative boundary conditions. The method is particularly well adapted to a reduced model which acts as a substitute for a detailed model with diminution of computation time. Although the reduced model is obtained with some kind of boundary conditions and usually functions within the latter, with these coupling fluxes, the reduced model can also be used in other conditions.

Nomenclature

$\mathbf{A}$ calculus matrix (dim.$N*N$)
b calculus vector (dim.N)
$\mathbf{C}$ matrix of conductances (dim.n_c*n_c) on Γ_c
$\mathbf{C}_a$ matrix of heat capacities (dim.$N*N$)
$\mathbf{F}$ diagonal matrix of n eigenvalues
$\mathbf{G}$ input matrix (dim.$n*(p-1)$)
$\mathbf{G}_0$ input matrix (dim.$n*p$)
$\mathbf{G}_c$ input matrix (dim.$n*n_c$)
$\mathbf{H}$ output matrix (dim.$N'*n$)
h heat transfer coefficient
$\mathbf{K}$ matrix of heat conductances (dim.$N*N$)
N order of DM
N' number of outputs
n order of RM
n_c number of nodes on Γ_c
p number of inputs
$\mathbf{R}$ matrix of radiative conductances (dim.n_c*nr_c) on Γ_c
$\mathbf{S}$ static matrix (dim.$N'*(p-1)$)
$\mathbf{S}_0$ static matrix (dim.$N*p$)
$\mathbf{S}_c$ static matrix (dim.$N'*n_c$)
T, $\dot{T}$ temperature vector and its derivative (dim.N)
u, $\dot{u}$ input vector and its derivative (dim.$p-1$)
U, $\dot{U}$ input vector and its derivative (dim.p)
x, $\dot{x}$ reduced state vector and its derivative (dim.n)
y output vector (dim.N').

Greek symbols

Γ boundary
Γ_c part of the boundary Γ to be treated
Δt time step
ε emissivity
θ temperature
Θ temperature vector
τ_i time constant
φ flux density
Φ vector of thermal stimulations (dim.N)
Φ_c vector of thermal stimulations (dim.N)
Ψ vector of thermal stimulations (dim.N).

Subscripts

1, 2 relative to both subsystems
c relative to the coupling
T transposition

* Corresponding author. E-mail: petit@let.univ-poitiers.fr

0017–9310/98 $19.00
PII: S0017-9310(98)00071-4

+ instant of computation
− previous instant of computation.

Abbreviations

BC boundary condition
DM detailed model
RM reduced model.

1. Introduction

The modelling of thermal processes governed by partial derivative equation generally needs a spatial discretization. Whatever the numerical method used (finite elements, control volumes, etc.) the resulting model can be written as a system of N differential equations in time. For a heat conduction problem (domain Ω and boundary Γ), a general formulation is:

$$\mathbf{C}_a \dot{T}(t) = \mathbf{K}T(t) + \Psi(t) \quad (1)$$

where t is the time, T (dim.N) is the vector of the N temperatures at the nodes of the discretization and $\dot{T}$ is its derivative with respect to time. For a linear problem, the matrices $\mathbf{C}_a$ (heat capacities) and $\mathbf{K}$ (thermal conductances) are constant. $\Psi(t)$ is the vector associated to heat generation and boundary conditions.

The model represented by equation (1) can be very large according to N. In order to reduce the simulation time, model reduction techniques coming from automatic works can be used [1–3]. In heat transfer problems, some of these methods have been used and improved [4–6] and new ones have been established [7–9]. In our approach, from a heat conduction problem such as equation (1), we have already shown how to obtain a reduced representation by an identification technique [8–14]. The obtained Reduced Model (RM) is expressed within a modal representation: the corresponding system of differential equations of order n ($n \ll N$) is moreover uncoupled. The n identified eigenvalues correspond to dominant modes, which are sufficient to describe the dynamic behaviour of the problem in a satisfactory manner. For example, it has been shown, in a 3-D case describing heat transfers within an electronic component, how the final simulation time has been reduced by a factor of 8000 [13, 14].

This kind of reduced model is, of course, particularly well adapted to real time applications or when a lot of simulations are needed. In this paper we want to show how to extend the field of the applications of such modelling to the coupling with other models, and more generally how to change boundary conditions on such models. A modal approach has already been used for the coupling problem [15]: the authors, after having reduced each subsystem with a modal formulation, calculate some coupling eigenmodes describing the connection of the subsystem. With regard to our work, the method is based on coupling fluxes, even if the reduced model has a modal formulation.

The basic problem is the following: let a diffusive domain Ω be with a boundary on which the boundary conditions (BC) are split into four parts (Fig. 1): BC of the first kind (Dirichlet) on Γ_1, BC of the second kind (Neumann) on Γ_2, BC of third kind (Fourier or Robin) on Γ_3. On the last part Γ_c is applied a BC of the second or third kind. On this part of the boundary, another type of BC corresponding to a coupling condition will be used afterwards.

Considering these hypothesis, eventually taking into account some heat generation into the domain, any modelling using a spatial discretization leads to the computation of a temperature vector at each time step. In this paper, the following cases will be considered successively:

- the temperature vector is computed with a classical detailed model (DM);
- all or part of this vector is computed with a reduced model (RM).

Now, we suppose that DM or RM is created. The aim is to keep such a model when applying new conditions on Γ_c:

- contact conditions (coupling with a fluid or a heat conduction model);
- radiative boundary conditions.

So, the procedure will permit the economy of not having to recreate a model corresponding to the new conditions on Γ_c. Let us emphasize that Γ_c must be relatively small compared to the whole boundary Γ. The coupling fluxes that will be applied on each node of Γ_c will give back a BC which was not included in the previous modelling. For this new interface, these fluxes will also be called correcting fluxes. In this paper, we explain how to build these correcting fluxes, at first on a classical detailed numerical model, and then, on the reduced model we have developed. Some examples with new BC (coupling

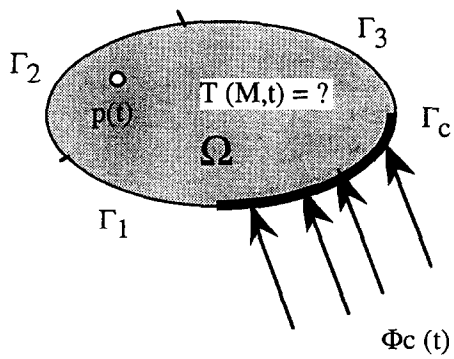

Fig. 1. The diffusive problem to treat, particularization of the boundary Γ_c.

of 2 RM, radiative BC) will be used to illustrate the method.

2. Modelling in order to couple

Note that the coupling method with the fluxes will be applied mainly when RMs are considered (i.e. coupling of 2 RMs or 1 RM and 1 DM). Nevertheless, as the equations have to be presented in a slightly different way, at first, we explain here the principle with a classic DM formulation (coupling of 2 DM's). Afterwards, the method is applied to RMs.

2.1. Correcting fluxes on a detailed model (DM)

Here is shown how a representation of a DM given by equation (1) and radiative to BC of the second kind on Γ_c, has to be modified in order to point out the interface variables of Γ_c: the latter will be treated specifically when the new interface condition will be applied. To particularize Γ_c, the flux vector Ψ is divided in two parts:

$$\Psi = \Phi + \Phi_c \tag{2}$$

Φ represents all the thermal stimulations applied to the system except on Γ_c, Φ_c is the contribution of Ψ which is applied only on Γ_c. Equation (1) is thus written:

$$\mathbf{C}_a \dot{T}(t) = \mathbf{K}T(t) + \Phi(t) + \Phi_c(t). \tag{3}$$

A time discretization is then necessary. Let $(+)$ be the superscript of the instant of calculation and $(-)$ the previous one. With an implicit Euler scheme, equation (3) becomes:

$$\frac{T^+ - T^-}{\Delta t} = \mathbf{C}_a^{-1}[\mathbf{K}T^+ + \Phi^+ + \Phi_c^+] \tag{4}$$

which leads to:

$$\mathbf{A}T^+ = b + b_c^+ \tag{5}$$

with:

$$\mathbf{A} = \frac{\mathbf{I}}{\Delta t} - \mathbf{C}_a^{-1}\mathbf{K}, \quad b = \frac{T^-}{\Delta t} + \mathbf{C}_a^{-1}\Phi^+$$

and $$b_c^+ = \mathbf{C}_a^{-1}\Phi_c^+ \tag{6}$$

where $\mathbf{I}$ is the identity matrix of order N. Vector b_c^+ includes all the fluxes Φ_c^+ applied on Γ_c. Vector b includes all other thermal stimulations Φ^+ as well as the temperature vector at the previous instant T^-. An other example of matrix $\mathbf{A}$ and vector b is given in Appendix 1 in the case of a centred scheme.

Remarks:

- to simplify the formulation, it will then be supposed that the n_c nodes on Γ_c have their temperatures listed at the beginning of vector T^+;
- hereafter it will be also supposed that matrix $\mathbf{A}$ can be inverted numerically: in fact, if the system is to large that can be an disadvantage. For the presentation, we need this inversion to separate the nodes on the boundary Γ_c. Later, for RM we will see that it is easier.

By taking into account this last remark, the solution of equation (5) is written as:

$$T^+ = \mathbf{A}^{-1}b + \mathbf{A}^{-1}\mathbf{C}_a^{-1}\Phi_c^+. \tag{7}$$

The vector Φ_c (dim.N) is such as: $\Phi_c^T = [\Phi_{cc}^T\, 0 \ldots 0]$ where Φ_{cc} (dim.n_c) is the real coupling vector. The term $\mathbf{A}^{-1}b$ contains all the past of the system as well as the contribution of the other solicitations (Φ^+). The term $\mathbf{A}^{-1}\mathbf{C}_a^{-1}\Phi_c^+$ corresponds to the contribution of what is applied on the boundary Γ_c at the instant $(+)$. It is precisely here that a BC other than a second kind one can be introduced through vector Φ_{cc}.

Let us write the n_c first lines of equation (7) in the following manner:

$$T_c^+ = (\mathbf{A}^{-1}b)_c + (\mathbf{A}^{-1}\mathbf{C}_a^{-1})_c\Phi_{cc}^+ \tag{8}$$

the subscript c indicates that the vectors and the matrices are limited to the n_c first lines.

For example, if an implicit scheme is used, it can be taken for Φ_{cc}:

- a contact BC: $\Phi_{cc}^+ = \mathbf{C}(\Theta_c^+ - T_c^+)$ (9)
- a radiative BC: $\Phi_{cc}^+ = \mathbf{R}(\Theta_c^{+4} - T_c^{+4})$. (10)

In these expressions, Θ is a temperature vector coming from another model, of which the n_c first components are facing the n_c first components of T (see Fig. 2 for a contact condition), $\mathbf{C}$ (dim.$n_c * n_c$) is a matrix expressing the new thermal conductances between the different nodes on both sides of Γ_c (linear transfer: contact with another solid or convection with a fluid). In the same way, $\mathbf{R}$ (dim.$n_c * n_c$) is a matrix expressing the radiative transfer with another system. Of course, a BC where both kinds of heat transfer appear is possible by combining equations (9) and (10).

If vector Φ_{cc}^+ is eliminated from equation (8) with (9) or (10), equation (8) constitutes then a system of n_c equations in two n_c unknowns (the node temperatures of both

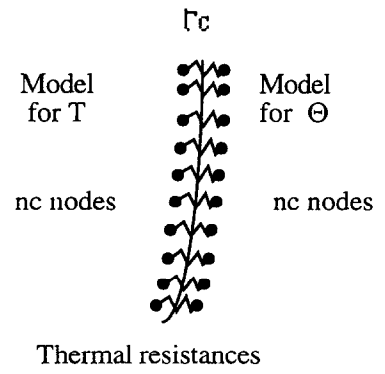

Fig. 2. Coupling conditions on Γ_c relatively to both models.

sides of Γ_c). To solve the problem, it is now necessary to add some information on the vector Θ. If we make the assumption that the Θ model is governed by similar equations as equations (7)–(10), the elimination of the coupling fluxes gives also another system of n_c equations in $2n_c$ unknowns: the association of these two systems then enables, at each time step, the calculation of the $2n_c$ values of temperature at the interface: T_c^+ and Θ_c^+. The method of resolution will be linear in the case of equation (9), and non linear in the case of equation (10).

Afterwards, the coupling flux vector is Φ_{cc}^+ calculated with (9) or (10). The rest of the temperature vector T^+ is calculated with (7). The same calculations are made in a similar way on the other side of Γ_c with Θ. Note that the full matrix $\mathbf{A}^{-1}\mathbf{C}_a^{-1}$ distributes the effect of Φ_c^+ on the temperatures of the nodes that are not located on Γ_c.

Remarks:

- in many cases—with a given fluid temperature $\Theta(t)$ for example—where the coupling is made with only one node, the local resolution of the system is of order n_c instead of $2n_c$;
- the proposed method enables the coupling two models. Note that there is no iteration between both models, the interface variables (temperatures and fluxes) being calculated in an explicit way at each time step;
- it is also possible to introducc a time-dependent heat transfer coefficient that will appear in matrix $\mathbf{C}$;
- it is always possible to approach a BC of first kind, or a perfect contact between two models, by setting numerically very important values for the terms of matrix $\mathbf{C}$;
- in the case of radiative BC, it is more interesting to solve a non-linear system of lower order on Γ_c than on the whole domain Ω. The calculation is faster and the convergence easier;
- remember that the method is worthwhile only if the number of nodes n_c of Γ_c is small compared to the total number of nodes N;
- the development which is presented here, needs the assumption that it is possible to invert two matrices ($\mathbf{A}$ for vector T and an equivalent matrix for Θ). These matrices can be large. Hereto, it will be seen how the method is particularly well adapted for reduced models, where the equivalent of $\mathbf{A}^{-1}$ is given in an explicit form.

2.2. *Correcting fluxes on a reduced model (RM)*

2.2.1. *The reduced model used*

The model reduction techniques have been used firstly in automatic sciences [1–3] and then used and developed in heat transfer modelling [4–9]. In our case, in previous works [8–14], we have already shown how to obtain, from a reference DM of order N, a RM of order n ($n \ll N$). The RM is expressed with a state space representation [17–18] using eigenmodes that have been identified. This identification is executed by using numerical simulations performed with DM.

Let us recall the main different steps:

- choice of N' nodes of the system (all the nodes can be taken, then $N' = N$);
- on each thermal input of the system, a unitary step is applied with DM and the N' time-dependent temperatures are stored;
- minimization of a quadratic criterion between these outputs and the ones coming from RM. The latter are expressed analytically and depend on the order n of RM and its parameters (the eigenmodes and the corresponding matrices);
- the resolution of this optimization problem leads to a state representation in the identified eigen basis:

$$\dot{x}(t) = \mathbf{F}x(t) + \mathbf{G}_0\dot{U}(t) \tag{11}$$

$$y(t) = \mathbf{H}x(t) + \mathbf{S}_0U(t) \tag{12}$$

where $x(t)$ is the reduced state vector whose dimension is very small compared to the reference DM ($n \ll N$), $\dot{x}$ is its derivative with respect to time, $\mathbf{F}$ is the diagonal matrix containing the dominant eigenmodes of the system, $U(t)$ (dim.p) is the input vector, $\dot{U}(t)$ its derivative and $y(t)$ (dim.N') is the estimation of the temperature vector made on the N' nodes selected in vector T. The appropriate dimensioned matrices $\mathbf{G}_0$ and $\mathbf{H}$, are called, respectively, the input and output matrices. $\mathbf{S}_0$ is the static matrix that points out the instantaneous asymptotic response $\mathbf{S}_0U(t)$. This output would be the one if the system had no heat capacity.

Remark:

A modal representation such equations (11)–(12), including the derivative with respect to time of the thermal solicitations and the static matrix is not often used in heat transfer modelling. In order to familiarize the reader with this, such a modelling is developed in the Appendix, in the case of DM—from equation (1).

Note that the reduction procedure leads to solve a differential equation system (11) of order $n \ll N$, that is moreover uncoupled (matrix $\mathbf{F}$ is diagonal). Note, also that the identified eigenvalues in $\mathbf{F}$ represent the dynamics: in heat diffusion, all these values (F_i) are real and negative, and correspond to inverses of time constants (τ_i):

$$F_i = -1/\tau_i. \tag{13}$$

The identification principle is recalled in Fig. 3.

2.2.2. *The reduced model (RM) with the correcting fluxes*

Let us suppose that RM given by equations (11)–(12) describes the temperature evolution of such a system represented in Fig. 1. In order to change BC on Γ_c, in a similar manner as what has been presented in Section 2.1, the thermal inputs included in vector $U(t)$ are divided in order to emphasize the contribution of $\Phi_c(t)$ as follows:

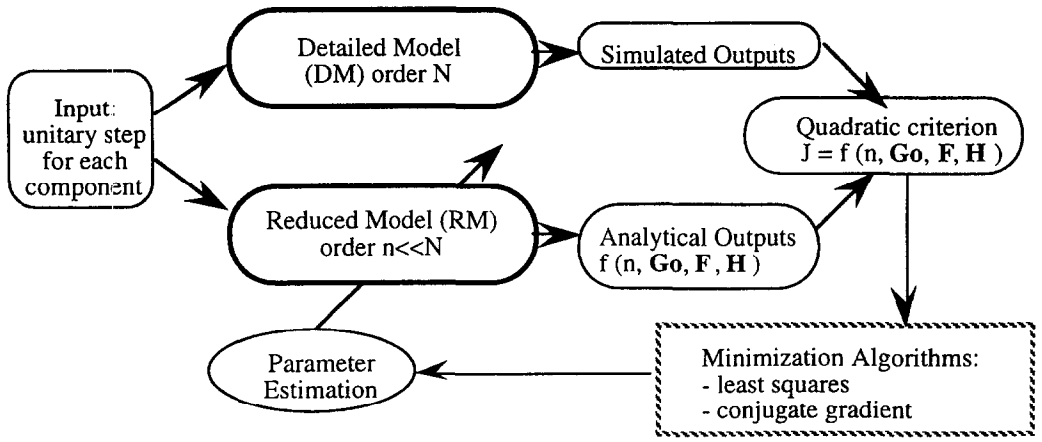

Fig. 3. Identification principle of the reduced model.

$$\dot{x}(t) = \mathbf{F}x(t) + \mathbf{G}\dot{u}(t) + \mathbf{G}_c\dot{\Phi}_c(t) \tag{14}$$

$$y(t) = \mathbf{H}x(t) + \mathbf{S}u(t) + \mathbf{S}_c\Phi_c(t). \tag{15}$$

The new input vector $u(t)$ and its derivative $\dot{u}(t)$ take into account all the thermal stimulations that do not act on Γ_c. They are, respectively, applied through the matrices **S** and **G**. The vector $\Phi_c(t)$ (dim.n_c) and its derivative $\dot{\Phi}_c(t)$ are applied on Γ_c, through the matrices $\mathbf{G}_c$ and $\mathbf{S}_c$.

2.2.2.1. Setting-up the static matrix $\mathbf{S}_c$. By using the reference DM in a steady state, a unit flux is applied successively on each of the n_c nodes, the other components of $u(t)$ are all null. Each of the n_c output vectors of DM then correspond to a column of the sought matrix. Note that this way of acting provides an exact static matrix: in fact $\mathbf{S}_c$ is the same in DM and RM.

2.2.2.2. Setting-up the input matrix $\mathbf{G}_c$. As it has been shown for the identification of the original RM, it is always possible to calculate $\mathbf{G}_c$ by studying the dynamic responses of the system when a step flux is made independently on each of the n_c nodes of Γ_c (as in Section 2.2.1). Such an approach becomes tedious if n_c is large. In practice, the dynamic response is studied when all the n_c nodes are stimulated simultaneously with only one step flux. At $t = 0$, the flux vector $\Phi_c = \underline{1}$ is applied. $\underline{1}$ is a vector (dim.n_c) including only ones. If the initial condition and the other inputs are null, the analytical solution of equations (14) and (15) is:

$$y(t) = [\mathbf{H}\exp(\mathbf{F}t)\mathbf{G}_c + \mathbf{S}_c]\underline{1}. \tag{16}$$

If the initial condition $y(0) = 0$ is applied, it ensues:

$$\mathbf{H}\mathbf{G}_c + \mathbf{S}_c = 0. \tag{17}$$

This relation represents a system of $N' * n_c$ equations (the term number of the matrix $\mathbf{S}_c$ previously determined) in $n * n_c$ unknowns (the term number of the matrix $\mathbf{G}_c$). With the assumption that $N' \geqslant n$, which is always the case in practice, it becomes an optimization problem and the matrix $\mathbf{G}_c$ can be calculated through the least squares method by:

$$\mathbf{G}_c = -(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{S}_c. \tag{18}$$

Remarks:

- $\mathbf{G}_c$ (dim.$n * n_c$) permits to distribute the influence of the variations of the correcting fluxes on the state vector $x(t)$ through the dynamics included in **F**;
- $\mathbf{G}_c$ depends on $\mathbf{S}_c$, which is exact and which expresses, in the steady case, the spatial distribution of Φ_c;
- $\mathbf{G}_c$ depends on **H** which has been identified with **F** in the dynamics procedure [8–14], recalled in Section 2.2.1. The same step flux has been used to stimulate all the nodes of Γ_c: they are then treated dynamically in the same way through matrices **F** and **H**. This single input has to be anticipated in the input vector $U(t)$ (dim.p) of the original RM [equations (11) and (12)];
- consequently, the vector $u(t)$ that includes all the other thermal informations is of dimension $p-1$. The matrices **G** and **S** are then, to within one column, the matrices $\mathbf{G}_0$ and $\mathbf{S}_0$;
- equation (17), that leads to equation (18), signifies that the instant $t = 0$ is particularized for the optimization problem: it is the moment when the step is applied. Other results have been obtained by minimizing criteria built on a integral from 0–∞ [8]. These results are less accurate;
- this reduction method enables us to obtain a RM on N' selected nodes set in $y(t)$ from the N components of the original vector $T(t)$ of DM. In order to couple, it is of course necessary to have at least the n_c nodes of Γ_c in $y(t)$.

Let us make a time discretization of equations (14) and (15). Using the analytical solution [17, 18] of equation (14) between the steps (+) and (−), it ensues:

$$x^+ = \exp(\mathbf{F}\Delta t)[x^- + \mathbf{G}(u^+ - u^-) + \mathbf{G}_c(\Phi_c^+ - \Phi_c^-)] \quad (19)$$

$$y^+ = \mathbf{H}x^+ + \mathbf{S}u^+ + \mathbf{S}_c\Phi_c^+. \quad (20)$$

This can be written as:

$$y^+ = y_0 + \mathbf{B}\Phi_c^+ \quad (21)$$

with:

$$y_0 = \mathbf{H}\exp(\mathbf{F}\Delta t)[x^- + \mathbf{G}(u^+ - u^-) - \mathbf{G}_c\Phi_c^-)] + \mathbf{S}u^+ \quad (22)$$

$$\mathbf{B} = \mathbf{H}\exp(\mathbf{F}\Delta t)\mathbf{G}_c + \mathbf{S}_c \quad (23)$$

y^+ is the solution at the time step (+), y_0 includes the past of the system at (−) as well as the contribution of the solicitation at (+) that are not applied on Γ_c, $\mathbf{B}$ is a matrix (dim.$N' * n_c$) that distributes over the whole vector $y(t)$, the effects of the applied fluxes included in Φ_c^+.

In the same manner as in Section 2.1, it will then be assumed hereto, that the n_c nodes on the boundary Γ_c have their temperature listed at the beginning of vector y^+. The relations allowing the change of BC are given with equivalent equations of (9) and (10):

- a contact BC: $\Phi_{cc}^+ = \mathbf{C}(\Theta_c^+ - y_c^+)$ (24)
- a radiative BC: $\Phi_{cc}^+ = \mathbf{R}(\Theta_c^{+4} - y_c^{+4})$. (25)

The subscript c and the matrices $\mathbf{C}$ and $\mathbf{R}$ have the same meanings as the ones introduced for equations (9) and (10). As for equation (8), the n_c first lines of (21) can be written as:

$$y_c^+ = y_{oc}^+ + \mathbf{B}_c\Phi_{cc}^+. \quad (26)$$

With the assumption that the model representing Θ is given in the same way as equation (26) for a RM or as equation (8) for a DM, the elimination of vector Φ_{cc}^+ provides a system of $2n_c$ equations in $2n_c$ unknowns (the temperatures on Γ_c). At each time step, the interface temperatures are calculated, then the flux vector Φ_{cc}^+ whose influence is distributed on the rest of the components of y^+ through equation (21).

Remarks:

- the proposed method is much easier in the case of RM than DM because it is not needed to invert the system matrix [see equation (7)]: in fact the relation (21) giving y^+ is explicit;
- of course, all the other remarks made for DM in Section 2.1 are valid, except for the final one;
- as it will be seen in the applications, the proposed method can integrate very rough changes of boundary conditions through the matrices $\mathbf{C}$ or $\mathbf{R}$ of equations (24) and (25).

3. Application to different cases

3.1. The coupling of two reduced models

3.1.1. The system under investigation

The 2-D conductive problem which is presented in Fig. 4 is composed with a concrete part (20 × 20 cm) and with an insulating one (20 × 8 cm). The supposed contact between both parts is nearly perfect with a thermal resistance of 10^{-7} K m W^{-1} for each element. On each of the faces AB, CD, BC, AD, are applied conditions of the third kind: $\theta_{AB}(t)$, $\theta_{CD}(t)$, $\theta_{BC}(t)$ and $\theta_{AD}(t)$ with the heat transfer coefficients (in W m^{-2} K^{-1}): $h_{AB} = 10$, $h_{CD} = 100$, $h_{BC} = 50$, $h_{AD} = 30$. The reference detailed model DM has been obtained from a classical control volume method: the spatial discretization includes $21 * 21 = 441$ nodes in the concrete, and $9 * 21 = 189$ nodes in the insulating, therefore a total of 630 nodes.

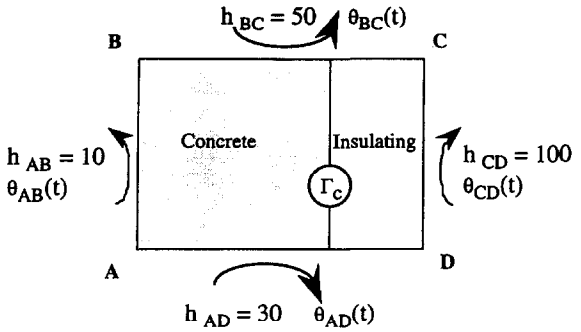

Fig. 4. The whole system to study.

3.1.2. The reduced models

Each of both subsystems is modelled separately with a DM (order 441 for the concrete and 189 for the insulating). For this study, a boundary condition of the third kind has been used on Γ_c where the contact will be made. The heat transfer coefficients are different (in W m^{-2} K^{-1}): $h_1 = 100$ and $h_2 = 80$, in relation to the temperature $\theta_c(t)$ (see Fig. 5). Therefore, each subsystem has four inputs. The identification procedure in order to obtain RM is then applied to each of both DMs. With the subscripts 1 for the concrete, and 2 for the insulating, according to equations (14) and (15), the reduced models are:

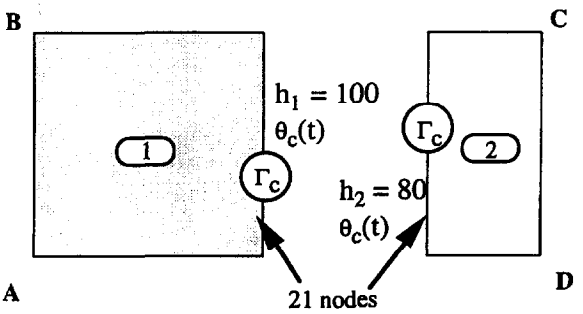

Fig. 5. The two uncoupled subsystems. The other BCs are the same as in Fig. 4.

$$\dot{x}_1 = \mathbf{F}_1 x_1 + \mathbf{G}_1 \dot{u}_1 + \mathbf{G}_{c1}\Phi_{c1}$$

$$\dot{x}_2 = \mathbf{F}_2 x_2 + \mathbf{G}_2 \dot{u}_2 + \mathbf{G}_{c2}\Phi_{c2}$$

$$y_1 = \mathbf{H}_1 x_1 + \mathbf{S}_1 u_1 + \mathbf{S}_{c1}\Phi_{c1} \quad \text{Concrete } (RM_1)$$

$$y_2 = \mathbf{H}_2 x_2 + \mathbf{S}_2 u_2 + \mathbf{S}_{c2}\Phi_{c2} \quad \text{Insulating } (RM_2). \qquad (27)$$

Remarks:

- RM1 has in $u_1(t)$ the three stimulations: $\theta_{AB}(t)$, $\theta_{BC}(t)$ and $\theta_{AD}(t)$, and similarly RM2 has in $u_2(t)$ the stimulations $\theta_{CD}(t)$, $\theta_{BC}(t)$ and $\theta_{AD}(t)$. The vectors $u_1(t)$ and $u_2(t)$ thus, regroup the real inputs of the whole;
- the vectors Φ_{c1} and Φ_{c2}, of dimension $n_c = 21$, corresponding to the nodes on each side of Γ_c, are 'fictitious' flux inputs on RM1 and RM2. They take into account the new coupling condition. They are calculated in a slightly different way than in equation (24) in order to take into account the initial BC of the third kind in each of the DM. For example, for RM1:

$$\Phi^+_{c1c} = \mathbf{C}(y^+_{2c} - y^+_{1c}) + h_2 y^+_{2c}. \qquad (28)$$

An equivalent relation is written for RM2.

- both RMs have been established for the whole temperature vector in both cases: the output vectors y_1 and y_2 are then vectors of dimension 441 and 189, respectively, corresponding to the two spatial discretizations.

For each RM, the identification method led us to build models of order five relatively to each input. Each matrix $\mathbf{F}_1$ and $\mathbf{F}_2$ then includes $5*4 = 20$ identified eigenvalues. To illustrate this point, the five eigenmodes relative to the input $\theta_c(t)$ are presented in Table 1, in the form of time constants [cf. equation (3)], for each RM.

Let us point out that each RM has its own dynamics: for example, the longest identified time constant is 1 h 31 min 50 s for the concrete and 1 h 2 min 1 s for the insulating. If we consider the whole system (concrete coupled with insulating) and if we extract the longest time constant from the corresponding DM of order 630, this latter is about 2 h 15 min. The coupling procedure of both RMs must then reproduce a quite different dynamics than the RM ones.

3.1.3. Simulations and comparisons

In order to test the reliability of the coupling, the system has been stimulated with very different inputs $\theta_{AB}(t)$, $\theta_{BC}(t)$, $\theta_{CD}(t)$ and $\theta_{AD}(t)$ allowing the creation of some inversions for the thermal gradient. In the time value range [0 – 100 000 s], the temperature variations (in °C) are given by (with t in s):

$\theta_{AB} = 7°C$ for $0 \leqslant t \leqslant 15\,000$ s,

$\theta_{AB} = 25°C$ for $t > 15\,000$ s

$\theta_{BC} = 25°C$ at $t = 0$, $\theta_{BC} = 11°C$ for $t > 0$

$\theta_{CD} = -9 * 10^{-4} t + 18$ for $0 \leqslant t \leqslant 2000$ s

$\theta_{CD} = 3.125\,10^{-4} t - 6.25$ for $t > 2000$ s

$\theta_{AD} = 1.4 * 10^{-4} t + 11 \quad \forall t.$

As these simulation durations are relatively long, a time step of 1000 s (16 min 40 s) has been chosen. Considering the identified time constants (see Table 1), the shortest ones cannot be represented with this time step. The used RMs are then of order three (instead of five) for each input.

Afterwards, the simulation results of the coupled RM is compared to the ones of DM of order 630. Among the 630 thermograms, nine comparisons of DM and the coupling RM1–RM2 are shown in Fig. 6: three thermograms (a, b, c) set inside the concrete, three thermograms (d, e, f) set inside the insulating, and three thermograms (g, h, i), at the interface on the concrete side.

Comments:

- as it can be seen, overall, the results are very satisfying. The coupling RM1–RM2 curves are always very close to the DM curves, even inside the concrete or the insulating or for the more sensitive nodes (d, e, f) at the interface;
- the computation times were 81 s for DM and less than one second for the coupling RM1–RM2, meaning a time gain in the order of 80: for DM a differential equation system of order 630 is treated, instead of two uncoupled systems of order 12 $(4 * 3)$ for RM1–RM2. Note that, in this application, the whole temperature vector is computed, this gain would be much better if the output vectors were smaller;
- the computation time to obtain RMs is made of two parts: the first part consists in the simulations of separated step inputs made with DM to obtain the thermograms to store (here about 320 s), and the second part is coming from the procedure of minimization is about 260 s. This time computation seems to be quite long (580 s) but they are made only one time. The obtained RMs can be used then for any solications and coupling;
- the computation for DM have been made with a direct method with preconditioning. An iterative method would never have been faster.

Table 1
The identified eigenvalues relative to input θ_c

τ_1	1 h 31 min 50 s	1 h 2 min 1 s
τ_2	20 min 6 s	20 min 59 s
τ_3	13 min 32 s	5 min 19 s
τ_4	4 min 55 s	37 s
τ_5	42 s	22 s

3.2. Coupling of a reduced model with radiative boundary conditions

After this study of linear conductive coupling between two reduced models, the subject under consideration

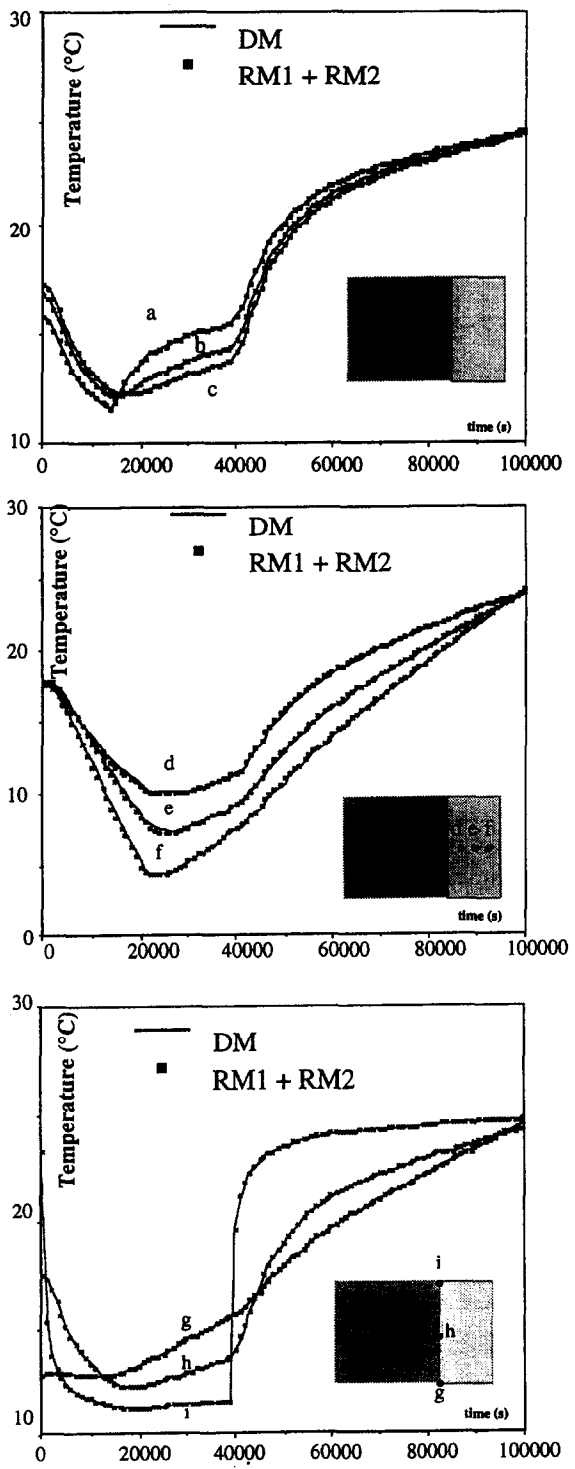

Fig. 6. Comparison between DM (order 630) and the coupling RM1 (order 12)–RM2 (order 12) for different nodes in the system.

here, is the use of such a RM (which has been identified with linear boundary conditions), with boundary radiative conditions by using this correcting flux method.

3.2.1. The studied system and the reference model

The example is a simple 2-D problem: a one metre square slab ABCD of which the heat capacity and the thermal conductivity are equal to one. The side AB is submitted to a flux density $\varphi(t)$. On the sides BC and CD, homogeneous boundary conditions of the second kind are imposed. On the side DA, corresponding to the boundary to be treated, two different cases will be considered (Fig. 7):

- case a: convective transfer ($h_{AD} = 5$ W m^{-2} K^{-1}) with the ambient environment being at $\theta(t)$;
- case b: same conditions as case a, upon which a radiative transfer is superimposed (emissivity $\varepsilon = 1$).

The reference DM for both cases has been obtained with the boundary element method [19, 20] which only requires a boundary mesh: 25 straight elements per side have been used (temperature and heat flux density are constant over each element). Thus the order of DM is 100.

3.2.2. The identified reduced model and its correction with fluxes

The method previously described enables to build a RM for case a, with the form of equations (9) and (10). For this study, the input vector $U(t)$ has then $p = 2$ components: $\varphi(t)$ and $\theta(t)$. The output vector $y(t)$ has been limited here to the $N' = 25$ elements of the side AD. Five eigenvalues have been identified for input φ, and six for input θ: the RM is then a model of order 11. The eigenvalues, under time constant form, are presented in Table 2. For comparison, the six first analytical time constants are also listed in this table.

Remarks:

- it can be noted that the longest time constant is identified on each input. The second and third also appear for the input θ;
- the other identified time constants correspond to an amalgamation of the rest of the spectrum;
- note also the dynamic uncoupling of each input which acts with its own eigenmodes according to the observed outputs.

The RM which has been created here, describes the system behaviour with heat transfer conditions relative to case a. It is now proposed, with this RM, to simulate the system behaviour when conditions of case b are applied.

In order to apply these new conditions, let us take equations (14) and (15). The side AD is supposed to exchange with the environment temperature $\theta(t)$. Φ_c is the correcting flux vector ($n_c = 25$), which is written, according to equation (25):

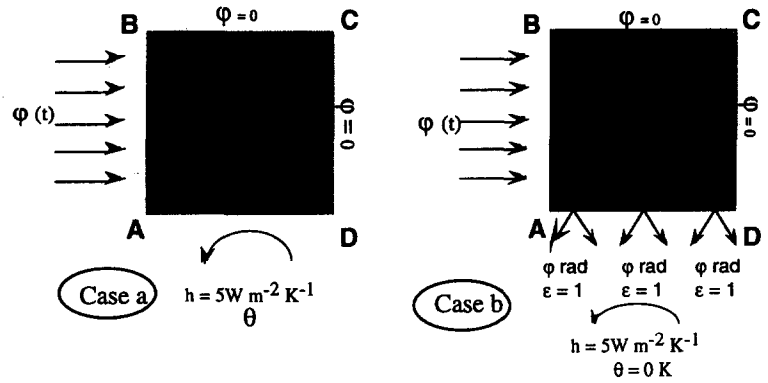

Fig. 7. Case a : linear convective transfer on AD. Case b : convective + radiative transfer.

Table 2
The different time constants (in s) : analytical and identified for each input

	The six first time constants obtained with an analytical method	The identified time constants for the input flux φ	The identified time constants for the input temperature θ
τ_1	0.579	0.575	0.577
τ_2	$6.14 * 10^{-2}$	$8.6 * 10^{-2}$	$6.25 * 10^{-2}$
τ_3	$2.09 * 10^{-2}$	$2.8 * 10^{-2}$	$2.17 * 10^{-2}$
τ_4	$1.02 * 10^{-2}$	$6.2 * 10^{-3}$	$6.46 * 10^{-3}$
τ_5	$5.97 * 10^{-3}$	$3.4 * 10^{-3}$	$1.02 * 10^{-3}$
τ_6	$3.09 * 10^{-3}$		$5.45 * 10^{-4}$

$$\Phi_c = \mathbf{R}(\theta^4 - y^4) \tag{29}$$

$\mathbf{R}$ (dim.25 * 25) is the corresponding radiative conductance matrix. An emissivity equal to one is applied to each element of the boundary AD.

3.2.3. Resolution of RM equations

3.2.3.1. The stationary state. At first, the initial stationary state is necessary. When vector Φ_c is introduced in equation (15) and, knowing that in the stationary case $x = 0$, we obtain :

$$y = \mathbf{S}u + \mathbf{S}_c\mathbf{R}(\theta^4 - y^4) \tag{30}$$

which can be written :

$$\mathbf{A}y^4 + y + b = 0 \tag{31}$$

with

$$\mathbf{A} = \mathbf{S}_c\mathbf{R}, \quad b = -\mathbf{S}u - \mathbf{S}_c\mathbf{R}\theta^4.$$

The Newton–Raphson method is used to solve this non linear equation [21].

3.2.3.2. The transient state. The time discretization of equation (14), associated to the correcting flux vector (29) leads to :

$$x^+ = x_0 - \exp(\mathbf{F}\Delta t)\mathbf{G}_c(\mathbf{R}y^{+4}) \tag{32}$$

with

$$x_0 = \exp(\mathbf{F}\Delta t)[x^- + \mathbf{G}(u^+ - u^-) + \mathbf{G}_c\mathbf{R}(\theta^{+4} - \theta^{-4} + y^{-4})] \tag{33}$$

where Δt is the time step between the calculation instant (+) and (−) the previous one. The term x_0 includes all the terms that are known at the instant (+).

Using equations (30) and (32) in the output equation (15), it ensues :

$$y^+ = \mathbf{H}x_0 - \mathbf{H}\exp(\mathbf{F}\,dt)\mathbf{G}_c\mathbf{R}y^{+4} + \mathbf{S}u^+ + \mathbf{S}_c\mathbf{R}\theta^{+4} - \mathbf{S}_c\mathbf{R}y^{+4}. \tag{34}$$

Grouping the terms concerning y^+ and y^{+4} we obtain :

$$\mathbf{A}y^{+4} + y^+ + b = 0 \tag{35}$$

with

$$\mathbf{A} = (\mathbf{H}\exp(\mathbf{F}\,dt)\mathbf{G}_c + \mathbf{S}_c)\mathbf{R} \tag{36}$$

$$b = -\mathbf{H}x_0 - \mathbf{S}u^+ - \mathbf{S}_c\mathbf{R}\theta^{+4}. \tag{37}$$

As in the case of the stationary state, the Newton–Raphson method enables the calculation of y^+ at each time step. If this last one is small, which is generally the case, the formulation can be linearized.

3.2.4. Simulations and results

In order to compare RM and DM in case b, the following inputs are used:

- for the initial steady state, $\theta = 0$ K and $\varphi = 1000$ W m^{-2};
- from this steady state, only $\varphi(t)$ will vary.

In Fig. 8 are presented the $\varphi(t)$ variation as well as six element temperature variations, chosen among the 25: the numbers 1, 5, 10, 15, 20 and 25. Element 1 is in contact with A and element 25 with D. For each element, three curves are presented:

- the above curve shows the results of simulations made with DM, without radiation (case a, $\varepsilon = 0$);
- the two other curves—which appear to be the same—show the results of case b ($\varepsilon = 1$) for DM and RM when taking into account the radiation through the correcting fluxes Φ_c.

Simulations have been made with 100 time steps of 0.02 s. As in the previous example, this choice of time induces the use of a RM of order three on each input when taking into account the identified time constants.

Comments:

- the comparison DM–RM for case a is not presented here: the curves are excellent, case b (more difficult to simulate) is more interesting. However, the DM curves are presented here essentially to emphasize the influence of the additive radiative conditions of case b;
- note that RM, identified in the linear context of case a, gives excellent results in case b when the correcting flux vector is used: the DM and RM curves cannot be distinguished from one another;
- these results are all the more interesting as the system is strongly stimulated: the temperature gradient along AD is strong and the corresponding radiative heat transfer is relative to a radiative temperature of 0 K. Even element 1, for which the temperature level is the most important (up to 500 K), has its temperature variation very well represented with RM;
- when comparing cases a and b, it can be noted that the correcting fluxes included in Φ_c are able to correct the output amplitude as well as their dynamics.

4. Conclusion

The method of correcting fluxes enables the division of a heat conduction problem into two parts:

—the first one relative to the diffusive domain where the boundary conditions are kept unchanged;
—the second one only relative to the boundary where the condition changes.

When associated to the technique of model reduction, this method is quite efficient and promising. It allows the use of an identified reduced model associated to boundary conditions which are very different from the ones used for identification. It has been shown here how different kinds of heat transfer couplings could be carried out: conduction–conduction, conduction–radiation. Of course, it is also possible to introduce a change in the heat transfer coefficient [9] which could be very interesting for some industrial applications such as heat exchangers for example.

In the frame of complex thermal systems, where the different kinds of heat transfers occur, the technique allows us to envisage some substructurations where the conduction parts would be calculated with reduced models. An application with parallel computing is also possible. The main disadvantage of the method is the fact that the part of the boundary to be coupled must not be too important compared to the whole boundary.

With this type of reduced model, we are currently working on the inverse heat conduction problem, consisting in the identification of the time varying inputs, from the knowledge of the outputs. This is a typical ill-posed problem, and the use of the reduced model acts as a regularization and gives some good results [16].

Furthermore, we also intend to work on the coupling of a heat conduction reduced model with a fluid mechanic model, as well as on the model reduction for the advection–diffusion problem.

Appendix 1: Time discretization for equation (3)

Let Δt be the time step. The current time step is noted with the subscript $(+)$, the previous one with the superscript $(-)$. The equation to treat is:

$$\mathbf{C}_a\dot{T}(t) = \mathbf{K}T(t) + \Phi(t) + \Phi_c(t).$$

For the centred scheme, by writing:

$$\dot{T} = \frac{T^+ - T^-}{\Delta t} \quad \text{and} \quad T = \frac{T^+ + T^-}{2}$$

in the equation, it ensues:

$$\mathbf{A} = \frac{2\mathbf{I}}{\Delta t} - \mathbf{C}_a^{-1}\mathbf{K},$$

$$b = \left(\frac{2\mathbf{I}}{\Delta t} + \mathbf{C}_a^{-1}\mathbf{K}\right)T^- + \mathbf{C}_a^{-1}(\Phi^+ + \Phi^- + \Phi_c^-)$$

and $b_c = \mathbf{C}_a^{-1}\Phi_c^+$

where $\mathbf{I}$ is the identity matrix of order N.

Note that b includes the coupling vector Φ_c, unlike the implicit scheme, but only at the previous instant of computation.

Appendix 2: Modal state equation with the static matrix and the input derivative vector

Let us take a classical system of N differential equations in time under the form:

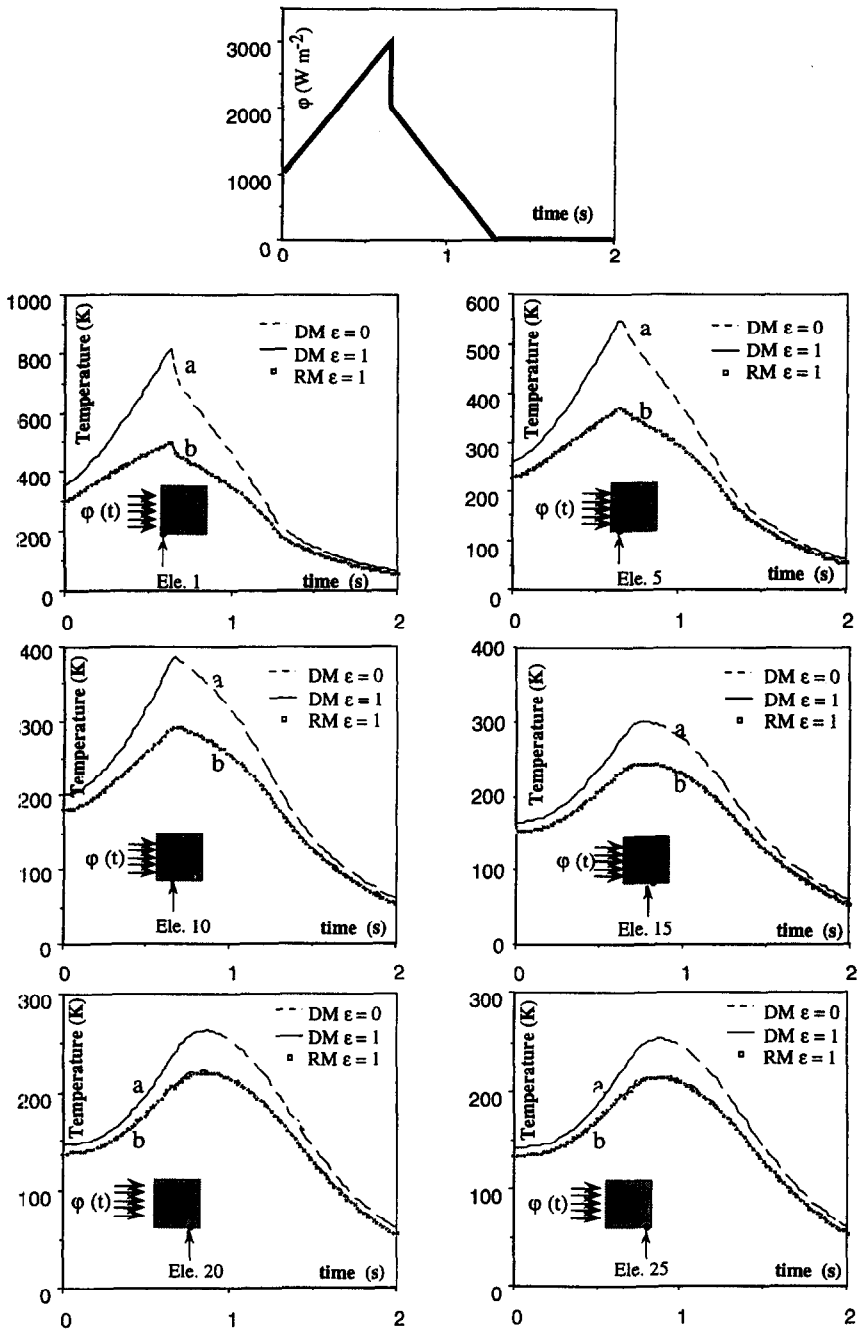

Fig. 8. Flux density $\varphi(t)$ vs. time and six of the 25 outputs: case a; DM without radiation ($\varepsilon = 0$), case b; comparison DM and RM with the correcting fluxes Φ_c.

$$\mathbf{C}_a\dot{T}(t) = \mathbf{K}T(t) + \Psi(t) \quad \text{(A1)}$$

that can be written as the state space representation:

$$\dot{T}(t) = \mathscr{A}T(t) + \mathscr{B}U(t) \quad \text{(A2)}$$

$\mathscr{A}$ (dim.$N * N$) is the state matrix such as: $\mathscr{A} = \mathbf{C}_a^{-1}\mathbf{K}$

$U(t)$ (dim.p) is the input vector which includes the thermal solicitations that are applied to the different nodes of $T(t)$ through matrix $\mathscr{B}$ (dim.$N * p$) such as: $\mathscr{B}$ $U(t) = \mathbf{C}_a^{-1}\Psi(t)$.

Often, an output equation is associated to equation (A2):

$$y(t) = \mathscr{C}T(t) \quad \text{(A3)}$$

which permits to keep, in the vector $y(t)$ (dim.N'), a selection (or a combination) of the components of $T(t)$ through the output matrix $\mathscr{C}$ (dim.$N' * N$).

Consider now a steady case governed by $U_s = Cst$. The corresponding solution T_s of equation (A2) is then

$$T_s = \mathscr{A}^{-1}\mathscr{B}U_s.$$

Thanks to this expression, we can introduce the new variable $T'(t) = -\mathscr{A}^{-1}\mathscr{B}U(t)$ that represents the quasi solution of equation (A2) if the system could react instantaneously to the vector $U(t)$.

In equations (A2) and (A3), let us introduce now the change of variables:

$$T(t) = T'(t) + X(t).$$

It ensues the new state space formulation:

$$\dot{X}(t) = \mathscr{A}X(t) + \mathscr{A}^{-1}\mathscr{B}\dot{U}(t) \quad \text{(A4)}$$

$$y(t) = \mathscr{C}X(t) - \mathscr{C}\mathscr{A}^{-1}\mathscr{B}U(t). \quad \text{(A5)}$$

$X(t)$ (dim.N) is a new state vector whose derivative is connected to the derivative of the input vector. The matrix $\mathscr{S} = -\mathscr{C}.\mathscr{A}^{-1}\mathscr{B}$ which appears in the output equation (A5) is called the static matrix (in fact, if $U = Cst$, $y = \mathscr{S}U$).

Note that the formulation (A4) and (A5) is equivalent to the original one (A1).

To obtain a modal formulation from (A4) and (A5), we make a new change of variables $X = \mathbf{M}x$, where $\mathbf{M}$ is the modal matrix (dim.$N * N$) which contains the N eigenvectors of matrix $\mathscr{A}$. It follows from this:

$$\dot{x}(t) = \mathscr{F}x(t) + \mathscr{G}\dot{U}(t) \quad \text{(A6)}$$

$$y(t) = \mathscr{H}x(t) + \mathscr{S}U(t) \quad \text{(A7)}$$

with: $\mathscr{F} = \mathbf{M}^{-1}\mathscr{A}^{-1}\mathbf{M}$ which is the diagonal matrix that contains the eigenvalues of $\mathscr{A}$, $\mathscr{G} = \mathbf{M}^{-1}\mathscr{A}^{-1}\mathscr{B}$, $\mathscr{H} = \mathscr{C}\mathbf{M}$.

Remarks:

- equations (A6) and (A7) are equivalent to (A1);
- equation (A6) is easier to integrate than equation (A1) because the components of $x(t)$ are uncoupled;
- numerous technics of reduction lean on the principle of decreasing the number of eigenvalues contained in equation (A6)

References

[1] Decoster M, Van Cauwenberghe AR. A comparative study of different reduction methods (Part 1-2) Journal A 1976;17(3):58–74, 125–35.

[2] Litz L. Order reduction of linear state space models via optimal approximation of the nondominant modes. North-Holland Publishing Company, Large Scale System 2. 1981, pp. 171–84.

[3] Eitelberg E. Comments on model reduction by minimizing equation error. IEEE Trans Automatic Control 1982;AC-27(4):1000–2.

[4] Shih TM, Skladnadany JT. An eigenvalue method for solving heat conduction problem. Numerical Heat Transfer 1983;6:409 22.

[5] El Khoury K, Neveu A. Analyse modale des systèmes thermiques en présence de transferts non réciproques. International Journal Heat Mass Transfer 1989;32:213–26.

[6] Ben Jaafar MT, Pasquetti R, Petit D. Model reduction for thermal diffusion: application of the Eitelberg, Marshall and aggregation methods to a heat transmission tube model. Int J for Numerical Methods in Engineering 1990;29:599–617.

[7] Oulefki A, Neveu A. Réduction par amalgame modal d'un modèle thermique. Journal de Physique 1993;3(2):303–20.

[8] Petit D. Réduction de modèles de connaissance et identification de modèles d'ordre reduit. Application aux Processus de Diffusion Thermique. Thèse d'état, Marseille, Avril, 1991.

[9] Hachette R. Réduction de modèle thermique par identification: utilisation pour des conditions aux limites variables. Couplage à un modèle détaillé. Thèse Université de Provence Marseille, 1995.

[10] Pasquetti R, Petit D. Analyse modale d'un processus de diffusion thermique: identification par thermographie infra-rouge. Int J Heat and Mass Transfer 1988;31:487–96.

[11] Petit D, Pasquetti R. Réduction de modèle par identification de modes dominants: application à un modèle bidimensionnel de diffusion thermique. Revue de Physique Appliquée 1990;29:599–617.

[12] Petit D. Systèmes thermiques diffusifs. Modélisation de leur comportement par identification de modes dominants. Revue générale de Thermique 1993;378:302–11.

[13] Petit D, Hachette R, Veyret D. Model reduction by identification: application to a heat transfer diffusion process. IASTED International Conference. Modelling, Identification and Control. 21–23 February 1994. Grindelwald, Suisse.

[14] Petit D, Hachette R, Veyret D. A modal identification method to reduce a high order model: application to heat conduction modelling. International Journal of Modelling and Simulation vol 17. April 1997 (to be published).

[15] Flament B, Bourquin FA. Neveu Synthèse modale: une méthode de sous-structuration dynamique pour la modélisation des systèmes thermiques linéaires. Int J Heat Mass Transfer 1993;36(6):1649–62.

[16] Petit D, Peruzzi S. Régularisation d'un problème inverse en conduction par utilisation d'un modèle réduit identifié. Congrès SFT. Juin 1996. Valenciennes, France.

[17] Eykhoff P. System Identification. John Wiley and Sons 1974–1987.

[18] De Larminat T, Thomas Y. Automatique des systèmes linéaires T1, T2, T3. Flammarion 1975–1977.
[19] Brebbia CA, Telles JCL, Wrobel LC. Boundary Elements Techniques. Springer-Verlag, 1984.
[20] Pasquetti R, Le Niliot C. Boundary element approach for inverse heat conduction problems: application to a bidimensional transient numerical experiment. Numerical Heat Transfer, Part B 1991;20:169–89.
[21] Press WH, Flannery BP, Teukolsky SA, Vetterling WT. Numerical Recipes. Cambridge University Press, 1990.